ARTIFICIAL VARIABLES :
Do They Provide Any Economic Information?

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I. Linear Programming has proved to be a very useful tool for decision making to solve linear optimization problems. It has been successfully applied to the problems of Product-Mix, Portfolio-Management, Advertising-Media, Distribution, Diet-Management, etc. To solve a Linear Programming Problem (LPP), Simplex method is still the best because it enables a decision maker to undertake the Sensitivity Analysis (i.e., "what if" analysis) which generates important economic information for decision making. Other methods, e.g. Karmarkar's Algorithm, may be computationally efficient but they are not subjected to the Sensitivity Analysis. Thus, they fail to provide the necessary information for decision making.

We introduce three kinds of variables into an LPP while solving it through simplex method : Slack variables, Surplus variables and Artificial variables. Each variable introduced into simplex method is subject to some economic interpretations. For instance, slack variable represents under-utilization of resources or idle capacity while surplus variable represents over-utilization of resources or excess capacity. Associated dual variables also have important economic interpretations, that is, shadow prices of resources. The importance of slack/surplus variables in the Sensitivity Analysis and in providing some significant economic information is well recognized in the literature (e.g. see Hadley (1987) and Levin et. al. (1982)). But, the literature fails to recognize the role of artificial variables in providing economic and significant information. It is believed that" . . . an artificial variable is only of value as a computational device." (Levin et. el. (1982), p.390). Even some authors suggest that once an artificial variable leaves the basis, there is no need to do computations under it. For instance, see Hadley's comments about artificial variables — "Since artificial vectors are never considered for re-entry into the basis once they have been removed, columns for these vectors need not be included in

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tableaux. Their inclusion would only lead to unnecessary computational efforts in the process of transforming the vectors." (Hadley (1987), p. 125). Thus, the role of artificial variables in decision-making is not properly recognized whether one uses BIG-M method or Two-Phase method. But, it is observed that the artificial variables do play a role in providing important economic information in some cases. 'What are such cases and how are they helpful in providing important economic information, is the subject matter of the present article.

II. An artificial variable (A) is introduced generally in constraints of the kinds "$\geq$" and "=" to obtain an initial basic solution for simplex method. Such a variable appears in "$\geq$" along with surplus variable which has -1 as coefficient in the constraint. (And, this is precisely for the reason that surplus variable is never taken into the basis of the initial basic solution). Since an artificial variable, A, always assumes a zero value in a feasible solution and it is not of any physical nature, it has, of its own, no economic interpretation. Hence, as such, it does not play any significant role in decision making. However, there exists a possibility that information available in C_j-Z_j row (also known as *N*et *E*valuation *R*ow, NER) under A could be of some use. The reason for anticipating this possibility is that such information may give values of the associated dual variables. But, in case of "$\geq$" constraint, one can easily verify that values in C_j-Z_j under surplus variables are identical to those under artificial variables *ignoring* M but with the opposite sign, assuming BIG-M method is used. Therefore, A has no role in "$\geq$" constraint.

III. Now, let us explore whether artificial variables have a role to play in "=" constraints. It is found that an artificial variable in a "=" constraint can be visualized as a slack/surplus variable which is *forced* to be zero. (That's why some authors use the term artificial slack variables for artificial variables. For example, see Loomba (1978). Therefore, A can be thought of one as behaving like a slack/surplus variable in a "=" constraint. Consequently, A can help in obtaining the value of associated dual variable which can be read from C_j-Z_j row of a simplex table under A ignoring M but read the value with opposite sign. We retain "+" and "-" because dual variable corresponding to a "=" constraint is an unrestricted variable. Thus, one may obtain important economic information and may undertake the Sensitivity Analysis.

We interpret the information taken from C_j-Z_j row of the final simplex table under A ignoring M and with opposite sign in the context of a Product-Mix problem. If such a value is negative, it indicates that lowering down corresponding b_i by one unit will increase total contribution by such value. And, if it comes to be positive, then lowering down corresponding b_i by one unit will decrease total contribution. Also, information in the column under A can be taken as marginal rates of substitution among basic variables with respect to the corresponding A. To illustrate what is said above, consider a numerical illustration below.

IV. **Numerical Illustration** : We are here considering a highly simplified Product Mix problem whose LPP model is :

$$\text{Max.Contribution} = 15x_1 + 20x_2 + 40x_3$$

Subject to

$2x_1 + 4x_2 + 6x_3 \leq 1800$ (Raw Material)
$4x_1 + 5x_2 + 8x_3 \leq 2400$ (Labour Hours) (M_1)
$3x_1 + 5x_2 + 10x_3 \leq 2400$ (Machine Hours)
$x_1, x_2, x_3 \geq 0$

where

$x_1 \equiv$ Quantity of Product#1 to be produced;
$x_2 \equiv$ Quantity of Product#2 to be produced;

and,

$x_3 \equiv$ Quantity of Product#3 to be produced.

Optimal Solution of M_1, using simplex method, is:

Max. Contribution = Rs. 10,500.00

$x_1 = 300$ $S_1 = 300$
$x_2 = 0$ $S_2 = 0$
$x_3 = 150$ $S_3 = 0$

Where S_i (i = 1,2,3) represents slack variable. According to the above solution, nothing of Product#2 will be produced. We further assume that due to some reason, the management is forced to produce 10 units of Product#2. Then, the required LPP model will be —

$$\text{Max. Contribution} = 15x_1 + 20x_2 + 40x_3$$

Subject to

$2x_1 + 4x_2 + 6x_3 \leq 1800$ (Raw Material)
$4x_1 + 5x_2 + 8x_3 \leq 2400$ (Labour Hours) (M_2)
$3x_1 + 5x_2 + 10x_3 \leq 2400$ (Machine Hours)
$x_2 = 10$
$x_1, x_2, x_3 \geq 0$

We obtain an optimal solution of M_2 and the relevant portion of the final table is reproduced below.

TABLE

C_j	BASIS	b_i	15	20	40	0	0	0	-M
			x_1	x_2	x_3	S_1	S_2	S_3	A
0	S_1	291.25							
15	x_1	293.75							
40	x_3	146.825							
20	x_2	10.00							
	Z_j	10481.25							
		C_j-Z_j	0	0	0	0	-1.875	-2.5	-M+1.875

We obtain -M+1.875 in C_j-Z_j row under A from the table above. Ignoring -M and taking the values with opposite sign, we get -1.875. -1.875 is interpreted as opportunity cost associated with the constraint $x_2 = 10$. It means that any increase in the production of Product#2 will decrease total contribution by Rs. 1.875. And, any decrease in the production of Product#2 is beneficial as it increases total contribution by Rs. 1.875. Hence, artificial variable column provides important economic information.

Next, we assume that the management decides to produce exactly 10 units of Product#1. Then, M_2 will become :

$$\begin{aligned}
&\text{Max. Contribution} = 15x_1 + 20x_2 + 40x_3 \\
&\text{Subject to} \\
&2x_1 + 4x_2 + 6x_3 \quad \leq 1800 \text{ (Raw Material)} \\
&4x_1 + 5x_2 + 8x_3 \quad \leq 2400 \text{ (Labour Hours)} \qquad (M_3) \\
&3x_1 + 5x_2 + 10x_3 \quad \leq 2400 \text{ (Machine Hours)} \\
&\qquad\qquad x_1 \quad = 10 \\
&x_1, x_2, x_3 \geq 0
\end{aligned}$$

We solve M_3 and give below the relevant portion of the final table of simplex procedure.

TABLE

C_j	BASIS	b_i	15	20	40	0	0	0	-M
			x_1	x_2	x_3	S_1	S_2	S_3	A
0	S_1	358.00							-0.20
0	S_2	464.00							-1.60
40	x_3	237.00							-0.30
15	x_1	10.00							1.00
	Z_j	9630.00							
		C_j-Z_j	0	0	0	0	0	-4	-M-3

From the above Table, we obtain 3 under A in C_j-Z_j row ignoring -M and with opposite sign. It indicates that a unit increase in the production of Product#1 will increase total contribution by Rs. 3/-; and, a unit decrease in the production of Product#1 will decrease total contribution by Rs. 3/-. Thus, the value of the dual variable corresponding to the constraint $x_1 = 10$ is Rs. 3/-. and, it is a useful and important economic information.

Likewise, we can illustrate the role of the column under A in simplex tableaux in the performance of the Sensitivity Analysis with respect to equality constraint. For instance, if b_i of the constraint $x_1 = 10$ in M_3 is increased to 11 then, the values of the basic variables will be changed according to the marginal rates of substitutions available under the column of A. The resultant values of the basic variables will be, then :

$$S_1 = 358.00 - 0.20 = 357.70$$
$$S_2 = 464.00 - 1.60 = 462.40$$
$$x_3 = 237.00 - 0.30 = 236.70$$
$$x_1 = 10.00 + 1.00 = 11.00$$

and, the resultant contribution will be Rs. 9630.00 + Rs. 3.00 = Rs. 9633.00.

V. **Conclusion:** It is wrong to believe that artificial variables are for *ONLY* computational ease. They may also provide useful and important economic information about shadow prices/opportunity costs that are associated with "=" constraints. One can also undertake the Sensitivity Analysis for "=" constraints using the information available in the column under A. Therefore, if an artificial variable is introduced into a "=" constraint, then one should not drop calculations under it when it leaves the basis.

NOTES AND REFERENCES

1. Hadley, G., *Linear Programming;* Narosa Publishing House, Delhi (1987).
2. Levin, I., Kirkpatrick, C.A. and Rubin, D.S., *Quantitative Approaches to Management;* 5th Ed.; McGraw Hill International Book Company (1982).
3. Loomba, N.P., *Management — A Quantitative Perspective;* Macmillan Publishing Inc., New York (1978).